Conformal deformation of warped products and scalar curvature functions on open manifolds

Man Chun LEUNG
Department of Mathematics,
National University of Singapore,
Singapore 119260
matlmc@nus.sg

April 1996

Abstract

We discuss conformal deformation and warped products on some open manifolds. We discuss how these can be applied to construct Riemannian metrics with specific scalar curvature functions.

KEY WORDS: scalar curvature, warped product, conformal deformation

1991 AMS MS Classifications: 53C20, 53A30

1. Introduction

In this paper we study scalar curvature functions on some open manifolds. A classification result of Kazdan and Warner (with an improvement by Bérard Bergery) states that if N is a compact n-manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories ([2] p. 125).

(A) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere; moreover, any metric with vanishing scalar curvature is Ricci-flat.

(C) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

Thus if the manifold N admits a Riemannian metric of positive scalar curvature, then it belongs to class (A). If N admits a Riemannian metric of zero scalar curvature and cannot admit a metric of positive scalar curvature, then it belongs to class (B). And any Riemannian metric on a manifold belongs to class (C) has scalar curvature negative somewhere. This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N. For noncompact manifolds, many important works have been done on the question of how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson [5] show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for examples, weakly enlargeable manifolds. Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([5], [6] p. 322). On the other hand, it is known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature [3]. It follows from the results of Aviles and McOwen [1] that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

By using conformal deformation, Ni and other authors have studied which functions on $\mathbf{R}^n$ are scalar curvatures of complete Riemannian metrics that are conformal to the Euclidean metric [8]. While in [10], Ratto, Rigoli and Véron give a rather detailed study of similar problem on a hyperbolic space using conformal deformation. In [7], we apply conformal deformation to study scalar curvatures on more general types of open manifolds. Due to large variety of structures on open manifolds, it is rather unclear how to consider the scalar curvature question on an arbitrary open manifold. We will mainly restrict ourselves to open manifolds that have compactifications.

Let $\overline{M}$ be a compact $(n+1)$-manifold with boundary ∂M and interior M. The boundary ∂M has finite number of connected components, and each component is a compact n-manifold without boundary. In this paper, we discuss the method of using warped products and conformal deformation to construct complete Riemannian metrics on M with specific scalar curvatures. By making use of the boundary, we can construct warped products at the ends of M. It is shown that if the boundary components belong to class (A) or (B), then M admits a complete Riemannian metric with positive scalar curvature outside a compact set. If a boundary component

belongs to class (C), then we can construct complete Riemannian metric with scalar curvature approaching zero near the end. We discuss restrictions of using warped products and conformal deformation to obtain complete Riemannian metric of positive scalar curvature outside a compact set. If a connected component of ∂M belongs to class (C), then we show that, under mild assumptions of warping functions, it is not possible to conformally deform warped product metrics to complete metrics on M with nonnegative scalar curvature outside a compact set (theorem 3.8). In section 4 we discuss the scalar curvature functions and conformal deformation of more general type of Riemannian metrics known as polar type Riemannian metrics . We discuss some restrictions of using polar type Riemannian metrics and conformal deformations to obtain nonnegative scalar curvature outside a compact set of M (theorem 4.18).

2. Boundary components in class (A) or (B)

Let (N, g) be a Riemannian manifold of dimenison n and let $f : (2, \infty) \to \mathbf{R}^+$ be a smooth function. The warped product of N and $(2, \infty)$ with warping function f is defined to be the Riemannian manifold $((2, \infty) \times N, g')$ with

$$g' = dt^2 + f^2(t) g \, . \tag{1.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature R of g' is given by the equation

$$R(t, x) = \frac{1}{f^2(t)} \{ R(g)(x) - 2n f(t) f''(t) - n(n-1)|f'(t)|^2 \} \tag{1.2}$$

for $t \in (2, \infty)$ and $x \in N$. In section 4 we discuss a more general formular for the scalar curvature of the Riemannian metric $dt^2 + f^2(x, t) g$, where f is a positive smooth function of $(2, \infty)$ and N. When f is a constant function, the metric g' in (1.1) is known as a conic metric. See [7] for a discussion of scalar curvature and conformal deformation of conic metrics. If we denote

$$u(t) = f^{\frac{n+1}{2}}(t) \, , \qquad t > 2 \, ,$$

then equation (1.2) can be changed into [4]

$$\frac{4n}{n+1} u'' + Ru - R(g) u^{\frac{n-3}{n+1}} = 0 \, . \tag{1.3}$$

Let $\overline{M}^n$ be a compact $(n+1)$-manifold with boundary ∂M and interior M. In this paper we assume that the boundary is nonempty and has connected components

$N_1, N_2, ..., N_k$. Each N_i is a compact n-manifold without boundary. A neighborhood of N_i in M is diffeomorphic to $(2, \infty) \times N_i$, $i = 1, 2, .., k$. The existence of a complete Riemannian metric on M with constant negative scalar curvature has been proved in [3]. In the present case we have a simplier proof.

Proposition 1.4. *If $n \geq 3$, then on M there is a complete metric of constant negative scalar curvature which is a product metric near infinity.*

Proof. On the compact manifold N_i, there exists a metric g_i of constant negative scalar curvature. Using the product metric on each $(2, \infty) \times N_i$ and extending to the whole M, we obtain a complete Riemannian metric g_1 on M with scalar curvature $R(g_1)$ being a negative constant outside a compact set. Let u be a nonnegative smooth function on M such that $u \equiv 1$ on $\partial M \times (2, b)$ for some $b > 2$, $u \equiv 0$ on $\cup_{i=1}^{k}(b+1, \infty) \times N_i$ and $| \bigtriangledown u| < C_o$ for some positive constant C_o independent on b. For b large enough, we have

$$\int_M (\frac{4n}{(n-1)} | \bigtriangledown u|^2 + R(g_1)u^2) dv_g < 0 \,.$$

Using a result of Aviles and McOwen [1], g can be conformally deformed into a complete Riemannian metric of constant negative scalar curvature. Furthermore, the conformal factor can be chosen to be equal to a positive constant outside a compact set. **Q.E.D.**

Corollary 1.5. [1] *Any negative smooth function on $\overline{M}$ is the scalar curvature of a complete metric on M.*

Proof. The function is a bounded negative function. Using the complete Riemannian metric constructed above, we can conformally deform it into a complete Riemannian metric with the prescribed scalar curvature. **Q.E.D.**

Consider the case when scalar curvature functions can be positive. For $1 \leq i \leq k$, if the manifold N_i admits a Riemannian metric of positive scalar cuvature, then the product metric

$$dt^2 + g_i$$

has positive scalar curvature on $(2\,, \infty) \times N_i$. If N_i admits a Riemannian metric of zero scalar curvature, then we let $u(t) = t^\alpha$ in (1.3), where $\alpha \in (0, 1)$ is a constant. We have

$$R = \frac{4n}{(n+1)} \alpha (1-\alpha) \frac{1}{t^2} > 0 \,, \qquad t > 2 \,. \tag{1.6}$$

Therefore we obtain the following.

Theorem 1.7. *For $n \geq 3$, let M be the interior of a compact $(n+1)$-manifold with boundary. Suppose that the boundary components are in class (A) or (B), then on M there is a complete Riemannian metric of positive scalar curvature outside a compact set*

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = 1/2$. And when $u = t^{\frac{1}{2}}$ we have

$$R = \frac{4n}{(n+1)} \frac{1}{4} \frac{1}{t^2}\,, \qquad t > 2\,.$$

We show that this is almost the best possible.

Lemma 1.8. *If $R(g) = 0$, then there are no positive solutions to the equation (1.3) with*

$$R(t) \geq \frac{4n}{(n+1)} \frac{c}{4} \frac{1}{t^2} \qquad for \quad t \geq t_o\,,$$

where $c > 1$ and $t_o > 2$ are constants.

Proof. Assume that

$$R(t) \geq \frac{4n}{(n+1)} \frac{c}{4} \frac{1}{t^2} \qquad \text{for} \quad t \geq t_o\,,$$

with $c > 1$. Equations (1.3) gives

$$t^2 u''(t) + \frac{c}{4} u \leq 0\,.$$

Let

$$u(t) = t^\alpha v(t)\,, \quad t \geq t_o\,,$$

where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u''(t) = \alpha(\alpha-1)t^{\alpha-2}v(t) + 2\alpha t^{\alpha-1}v'(t) + t^\alpha v''(t)\,.$$

And we obtain

$$t^\alpha v(t)[\alpha(\alpha-1) + \frac{c}{4}] + 2\alpha t^{\alpha+1}v'(t) + t^{\alpha+2}v''(t) \leq 0\,. \tag{1.9}$$

Let δ be a positive constant such that $\delta^2 = (c-1)/4$. Then we have

$$\alpha(\alpha-1) + \frac{c}{4} = (\alpha - \frac{1}{2})^2 + \frac{c-1}{4} \geq \delta^2\,.$$

Then δ is a constant independent on α. (1.9) gives

$$2\alpha t v'(t) + t^2 v''(t) \leq -\delta^2 v(t)\,. \tag{1.10}$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta < 1$, that is, $\alpha < 1/2$. Then (1.10) becomes

$$(t^\beta v'(t))' \le -\frac{\delta^2 v(t)}{t^{2-\beta}}\,.$$

Upon integration we have

$$t^\beta v'(t) - \tau^\beta v'(\tau) \le -\int_\tau^t \frac{\delta^2 v(s)}{s^{2-\beta}}ds\,, \qquad t > \tau > t_o\,. \tag{1.11}$$

If $v'(\tau) \le 0$ for some $\tau > t_o$, then (1.11) impliest that

$$t^\beta v'(t) \le -C$$

for some positive constant C. We have

$$v(t) \le v(\tau) - \int_\tau^t \frac{C}{s^\beta}ds = v(\tau) - \frac{t^{1-\beta}}{1-\beta}|_\tau^t \to -\infty\,,$$

as $\beta < 1$. Hence $v(t) < 0$ for some t, contradicting that $u(t) > 0$ for all $t \ge t_o$. Thus we have $v'(t) > 0$ for all $t > t_o$. (1.11) implies that

$$\tau^\beta v'(\tau) - \int_\tau^t \frac{\delta^2 v(s)}{s^{2-\beta}}ds \ge 0$$

for all $t > \tau > t_o$. As $v'(t) > 0$ for all $t > t_o$, we have

$$\tau^\beta v'(\tau) \ge v(\tau)\int_\tau^t \frac{\delta^2}{s^{2-\beta}}ds = v(\tau)\frac{1}{s^{1-\beta}}[-\frac{\delta^2}{1-\beta}]|_\tau^t\,.$$

Let $t \to \infty$ we have

$$\tau^\beta v'(\tau) \ge \frac{v(\tau)}{\tau^{1-\beta}}\frac{\delta^2}{1-\beta}\,.$$

Or after changing the parameter we have

$$\frac{v'(t)}{v(t)} \ge \frac{1}{t}\frac{\delta^2}{1-\beta}\,, \quad t > t_o\,.$$

Choosing $\alpha < 1/2$ close to $1/2$ so that $\beta < 1$ is close to 1 and using the fact that δ is independent on α or β, we have

$$\frac{v'(t)}{v(t)} \ge \frac{N}{t}$$

for a big integer $N > 2$. This gives

$$v(t) \ge Ct^N\,, \quad t > t_o\,,$$

where C is a positive constant. (1.11) implies that

$$t^\beta v'(t) \le \tau^\beta v'(\tau) - \int_\tau^t \frac{C\delta^2 s^N}{s^{2-\beta}}ds \to -\infty \quad \text{as} \quad t \to \infty\,.$$

Thus $v'(t) < 0$ for t large. **Q.E.D.**

In particular, if $R(g) = 0$, then using warped product it is impossible to obtain a Riemannian metric of uniformly positive scalar curvature. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is positive but goes to zero at infinity. It can also be shown that one cannot conformally deform the metric

$$dt^2 + t^{\frac{2}{n+1}} g$$

into a complete metric of uniformly positive scalar curvature outside a compact set. For simplicity we assume that the boundary of $\overline{M}$ is connected. The case where $\partial\overline{M}$ has more than one connected component is similar.

Theorem 1.12 *For $n \geq 3$, let $\overline{M}$ be a compact $(n+1)$-manifold with boundary N. Suppose that N admits a Riemannian metric g of zero scalar curvature. Let g' be a complete Riemannian metric on M with*

$$g'(t, x) = dt^2 + t^{\frac{2}{n+1}} g(x) \quad \text{on} \quad (2, \infty) \times \partial M\,.$$

Then the Riemannian metric g' cannot be conformally deformed in a complete Riemannian metric of uniformly positive scalar curvature outside a compact set.

Proof. For the metric $g' = dt^2 + f^2(t) g$ the Laplacian is given by

$$\Delta_{g'}\, u = \frac{\partial^2 u}{\partial t^2} + \frac{n f'(t)}{f(t)} \frac{\partial u}{\partial t} + \frac{1}{f^2(t)} \Delta_g u\,, \tag{1.13}$$

where Δ_g is the Laplacian for the Riemannian metric g. Let $g'' = u^{4/(n-1)} g'$ be a conformal deformation of g', where u is a positive smooth function. Assume that g'' is complete and the scalar curvature of g'', $R_{g''} \geq c^2$ outside a compact set, where c is positive constant. Using (1.2) and the conformal scalar curvature eqaution [7], we have

$$\frac{\partial^2 u}{\partial t^2} + \frac{n f'(t)}{f(t)} \frac{\partial u}{\partial t} + \frac{1}{f^2(t)} \Delta_g u - \frac{1}{f^2(t)} \frac{n-1}{4n} \{2n f(t) f''(t) + n(n-1) |f'(t)|^2\} u \leq -c^2 \frac{(n-1)}{4n} u^{\frac{n+3}{n-1}}$$

for all $t \geq t'$, where $t' > 2$ is a constant. Fix a value $t \geq t'$ and integrate the above inequality with respect to the fixed Riemannian manifold (N, g), we have

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\Big(\int_N u(x,t) dv_g\Big) + \frac{n f'(t)}{f(t)} \frac{\partial}{\partial t}\Big(\int_N u(x,t) dv_g\Big) \\ &- \frac{1}{f^2(t)} \frac{n-1}{4n} \{2n f(t) f''(t) + n(n-1)|f'(t)|^2\}\Big(\int_N u(x,t) dv_g\Big) \leq -c^2 \frac{(n-1)}{4n} \int_N u^{\frac{n+3}{n-1}} dv_g\,, \end{aligned} \tag{1.14}$$

where we have used Green's identity. For $t \geq t'$, let

$$U(t) = \int_N u(x,t) dv_g\,.$$

Using the Hölder inequality we have

$$\int_N u^{\frac{n+3}{n-1}} dv_g \geq \frac{[\int_N u dv_g]^{\frac{n+3}{n-1}}}{[\mathrm{Vol}(N,g)]^{\frac{4}{n+3}}} \, .$$

Thus we have

$$\frac{d^2U}{dt^2} + \frac{nf'(t)}{f(t)}\frac{dU}{dt} - \frac{1}{f^2(t)}\frac{n-1}{4n}\{2nf(t)f''(t) + n(n-1)|f'(t)|^2\}U \leq -\epsilon^2 U^{\frac{n+3}{n-1}} \, , \tag{1.15}$$

where ϵ is a positive constant given by

$$\epsilon^2 = \frac{c^2(n-1)}{4n[\mathrm{Vol}(N,g)]^{\frac{4}{n+3}}} \, .$$

In this case $f(t) = t^{1/(n+1)}$ for $t > 2$, we have

$$U'' + \frac{n}{n+1}\frac{U'}{t} - \frac{n-1}{4(n+1)}\frac{U}{t^2} \leq -\epsilon^2 U^{\frac{n+3}{n-1}} \, . \tag{1.16}$$

Let $U(t) = t^\alpha v(t)$ for $t > t'$, where α is a constant to be determined later and $v(t)$ is a positive smooth function. We have

$$U''(t) = \alpha(\alpha-1)t^{\alpha-2}v(t) + 2\alpha t^{\alpha-1}v'(t) + t^\alpha v''(t) \, .$$

Equation (1.16) gives

$$t^{\alpha-2}v[\alpha^2 - \frac{1}{n+1}\alpha - \frac{n-1}{4(n+1)}] + t^{\alpha-1}v'[\frac{n}{n+1} + 2\alpha] + t^\alpha v''(t) \leq -\epsilon^2 t^{\frac{n+3}{n-1}\alpha} v^{\frac{n+3}{n-1}} \, . \tag{1.17}$$

The quadratic term

$$q(\alpha) = \alpha^2 - \frac{1}{n+1}\alpha - \frac{n-1}{4(n+1)}$$

has zeros at

$$\frac{1}{2}(\frac{1}{n+1} \pm \sqrt{\frac{1}{(n+1)^2} + \frac{n-1}{n+1}}) \, .$$

For

$$\alpha < \frac{1}{2}(\frac{1}{n+1} - \sqrt{\frac{1}{(n+1)^2} + \frac{n-1}{n+1}}) \, ,$$

we have $q(\alpha) > 0$. If we take $\alpha = -(n-1)/2$, then

$$\alpha - 2 - \frac{n+3}{n-1}\alpha = 0 \, .$$

From (1.17) we have

$$tv'(\frac{n}{n+1} + 2\alpha) + t^2 v'' \leq -\epsilon^2 v^{\frac{n+3}{n-1}} \, .$$

Or

$$(1.18) \qquad (t^{\frac{n}{n+1}+2\alpha}v')' \le -\epsilon^2 t^{\frac{n}{n+1}+2\alpha-2} v^{\frac{n+3}{n-1}}\,.$$

Integrating both sides of (1.18) we obtain

$$(1.19) \qquad t^{\frac{n}{n+1}+2\alpha}v'(t) \le t_o^{\frac{n}{n+1}+2\alpha}v'(t_o) - \epsilon^2 \int_{t_o}^{t} s^{\frac{n}{n+1}+2\alpha-2} v(s)^{\frac{n+3}{n-1}} ds\,.$$

If there exists a $t_o > t'$ such that $v'(t_o) < 0$, then we have

$$t^{\frac{n}{n+1}+2\alpha}v'(t) \le -c^2$$

for all $t > t_o$, where c is a positive constant. Thus

$$v'(t) \le -c^2 t^{n-1-\frac{n}{n+1}}$$

for all $t > t_o$. Therefore we have $v(t) \le 0$ for t large, which is a contracdiction. Hence $v'(t) \ge 0$ for all $t > t'$. Then the inequality in (1.19) gives

$$(1.20) \qquad \tau^{\frac{n}{n+1}+2\alpha}v'(\tau) \ge \epsilon^2 \int_{\tau}^{t} s^{\frac{n}{n+1}+2\alpha-2} v(s)^{\frac{n+3}{n-1}} ds\,,$$

where $t > \tau > t'$. As $v'(t) \ge 0$, we have

$$\tau^{\frac{n}{n+1}+2\alpha}v'(\tau) \ge \epsilon^2 v(\tau)^{\frac{n+3}{n-1}} \int_{\tau}^{t} s^{\frac{n}{n+1}+2\alpha-2} ds\,.$$

As $\alpha = -(n-1)/2$ and $n \ge 3$, we have

$$\frac{n}{n+1} + 2\alpha - 2 < -1\,.$$

Therefore after integration and let $t \to \infty$, we have

$$v'(\tau) \ge c_1(\frac{1}{\tau})v(\tau)^{\frac{n+3}{n-1}} \ge c_2 \frac{v(\tau)}{\tau}$$

for all $\tau > 2$, where c_1 and c_2 are positive constants. We have made use of the fact that $v'(t) > 0$ for all $t > 2$ implies that $v(t)$ is bounded from below by a positive constant. Thus

$$v(t) \ge Ct^{c_2}$$

for all $t > 3$, where C is a positive constant. Substitute into the right hand side of (1.20) gives

$$\tau^{\frac{n}{n+1}+2\alpha}v'(\tau) \ge C_1 \int_{\tau}^{t} s^{\frac{n}{n+1}+2\alpha-2+c_2\frac{n+3}{n-1}} ds\,,$$

for $t > \tau > 3$. As in above, after integration we obtain

$$v'(\tau) \ge c_3 \frac{v(\tau)}{\tau^\delta} \ge C_2 \frac{v(\tau)}{\tau}$$

where $\delta < 1$ is a positive constant, c_3 and C_2 are positive constant. Furthermore, we may assume that

$$C_1 \ge \frac{n}{n+1} + 2\alpha - 2$$

for all t large enough. Thus

$$v(t) \ge C_3 t^{C_2}$$

for all t large enough, where C_3 is a positive constant. Substitute into the first inequality in (1.20) gives

$$\tau^{\frac{n}{n+1}+2\alpha} v'(\tau) \ge C_4 \int_\tau^t ds \tag{1.21}$$

for all τ large enough, where C_4 is a positive constant. But this is impossible as the right hand side of (1.21) tends to infinity as $t \to \infty$. **Q.E.D.**

3. Boundary components in class (C)

In this section we assume that at least one of the boundary components of $\overline{M}$, namely N_1, belongs to class (C). Then any Riemannian metric g on N_1 would have the scalar curvature negative somewhere. Take a Riemannian metric g_1 on N_1 with $R(g_1) = -n(n-1)$. Then equations (1.3) becomes

$$\frac{4n}{n+1} u'' + n(n-1) u^{\frac{n-3}{n+1}} + Ru = 0\,. \tag{3.1}$$

Lemma 3.2. *Assume that $R \in C^\infty([2,\infty))$ is a negative function such that $R \ge -a^2$ for some positive constant a and*

$$R(t) \le -\frac{C}{t^\alpha} \quad \text{for} \quad t \ge t_o\,,$$

where $t_o > 2$, C and $\alpha \le 2$ are positive constants. If $\alpha = 2$, we assume that $C > n(n-1)$. Then equation (3.1) has a positive solution on $(2, \infty)$.

Proof. If $\alpha < 2$, then we let $u_+ = c_+ + t^m$ and $u_- = c_-$, where c_+, c_- and m are positive numbers. If we take c_+ and m large enough and take c_- small, then we have

$$\frac{4n}{n+1} u_+" + n(n-1) u_+^{\frac{n-3}{n+1}} + Ru_+ \le 0\,,$$

$$\frac{4n}{n+1}u_-" + n(n-1)u_-^{\frac{n-3}{n+1}} + Ru_- \geq 0\,.$$

By the upper and lower solution method, we obtain a positive solution (c.f. [7]). In case $\alpha = 2$ and $C > n(n-1)$, we may take $u_+ = C_+ t^{(n+1)/2}$, where C_+ is a positive constant. Then

$$\begin{aligned} & \frac{4n}{n+1}u_+'' + n(n-1)u_+^{\frac{n-3}{n+1}} + Ru_+ \\ \leq \; & C_+ n(n-1)t^{\frac{n-3}{2}} + n(n+1)C_+^{\frac{n-3}{n+1}}t^{\frac{n-3}{2}} - C_+[n(n-1)+\epsilon]t^{\frac{n-3}{2}} \leq 0\,, \end{aligned}$$

if we take C_+ to be large enough. Here $\epsilon = C - n(n-1) > 0$ is a positive constant. Take u_- to be a small positive constant. In this case, we obtain a positive solution as in above. **Q.E.D.**

In the above lemma, when $\alpha = 2$ and $C \geq n(n-1)$, we have the following.

Lemma 3.3. *Suppose that N_1 belongs to class (C). Let g be a Riemannian metric on N_1. On the end $(2,\infty) \times N_1$, there does not exist a warped product metric*

$$g' = dt^2 + f^2(t)g$$

with scalar curvature

$$R \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N_1$ and $t > t_o > 2$, where t_o is a constant.

Proof. Assume that we can find a warped product metric on $(2,\infty)\times N_1$ with

$$R \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N_1$ and $t > t_o > 2$. We may assume that the scalar curvature of g is equal to $-\kappa^2$ at $x_o \in N_1$, where κ is a positive constant. With $u(t) = f^{\frac{n+1}{2}}(t)$ and at x_o, by (1.3) we have

$$\frac{4n}{n+1}\frac{u''}{u} + \frac{\kappa^2}{u^{\frac{4}{n+1}}} \leq \frac{n(n-1)}{t^2}\,. \tag{3.4}$$

In particular

$$\frac{u''}{u} \leq \frac{(n+1)(n-1)}{4t^2}\,.$$

That is,

$$t^{\frac{n+1}{2}}u'' \leq \frac{(n+1)(n-1)}{4}t^{\frac{n-3}{2}}u\,.$$

Consider the inequality

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2}$$

for $t > t_o > 2$, where $C \geq 1$ is a constant. Let $\varepsilon > 1$ be a constant such that $\varepsilon(\varepsilon - 1) = C$. Then we have

$$t^\varepsilon u''(t) \leq \varepsilon(\varepsilon - 1)t^{\varepsilon-2}u(t)\,.$$

Upon integration from $t_1 \geq t_o$ to $t > t_1$, and using integration by parts twice, we obtain

$$t^\varepsilon u'(t) - \varepsilon t^{\varepsilon-1}u(t) - t_1^\varepsilon u'(t_1) + \varepsilon t_1^{\varepsilon-1}u(t_1) + \varepsilon(\varepsilon-1)\int_{t_1}^t s^{\varepsilon-2}u(s)ds \leq C\int_{t_1}^t s^{\varepsilon-2}u(s)ds\,.$$

Therefore we have

$$t^\varepsilon u'(t) - \varepsilon t^{\varepsilon-1}u(t) \leq t_1^\varepsilon u'(t_1) - \varepsilon t_1^{\varepsilon-1}u(t_1)\,. \tag{3.5}$$

If there is a number $t_1 \geq t_o$ such that $u'(t_1) \leq 0$, then we have

$$t^\varepsilon u'(t) - \varepsilon t^{\varepsilon-1}u(t) \leq 0\,.$$

This gives

$$(\ln u(t))' \leq \varepsilon(\ln t)'\,.$$

Hence

$$u(t) \leq ct^\varepsilon$$

for all $t > t_1$, where c is a positive constant. If $u'(t) > 0$ for all $t \geq t_o$, then $u(t) \geq c'$ for some positive constant c'. Let C be a positive constant such that

$$t_1^\varepsilon u'(t_1) - \varepsilon t_1^{\varepsilon-1}u(t_1) \leq C\,,$$

then (3.5) gives

$$t^\varepsilon u'(t) - \varepsilon t^{\varepsilon-1}u(t) \leq C$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \leq \frac{\varepsilon}{t} + \frac{C}{u(t)t^\varepsilon} \leq \frac{\varepsilon}{t} + \frac{C}{ct^\varepsilon}\,.$$

Integrating from t_1 to t we have

$$\ln\frac{u(t)}{u(t_1)} \leq \varepsilon\ln(\frac{t}{t_1}) + \frac{C}{c't_1^{\varepsilon-1}} \leq \varepsilon\ln(\frac{C't}{t_1})\,,$$

as $\varepsilon > 1$. Here C' is a positive constant such that $\ln C' \geq C/(c't_1)$. Hence we again obtain the inequality

$$u(t) \leq bt^\varepsilon$$

for some positive constant b and for all $t \geq t_1$. Thus we find a constant $c > 0$ such that

$$u(t) \leq ct^{\varepsilon} \tag{3.6}$$

for all $t \geq t_1$. In case $C = (n+1)(n-1)/4$ we take $\varepsilon = (n+1)/2$, we have

$$u(t) \leq ct^{\frac{n+1}{2}}\,.$$

Then

$$\frac{\kappa^2}{u^{\frac{4}{n+1}}} \geq \frac{c'}{t^2}\,,$$

where c' is a positive constant. Hence (3.4) gives

$$\frac{u''}{u} \leq \frac{(n+1)(n-1)-\delta}{4t^2}\,,$$

where $\delta > 0$ is a costant. Similarly we have

$$u(t) \leq ct^{\frac{n+1}{2}-\delta'}$$

and

$$\frac{\kappa^2}{u^{\frac{4}{n+1}}} \geq \frac{c''}{t^{2-\epsilon}}\,.$$

for some positive constants δ', ϵ and c''. Thus (3.4) gives

$$u''(t) \leq 0$$

for t large and hence $u(t) \leq Ct$ for some contant $C > 0$. From (3.4) we have

$$\frac{u''(t)}{u(t)} \leq -\frac{\kappa^2}{(Ct)^{\frac{4}{n+1}}} + \frac{n(n-1)}{t^2} \leq -\frac{c}{t}$$

for t large enough, as $n \geq 3$. Here c is a positive constant. We have

$$u'(t) - u'(t') \leq -c\int_{t'}^{t} \frac{u(s)}{s}ds\,, \qquad t > t_1\,.$$

If $u'(t') \leq 0$ for some t', then $u'(t) \leq -c_1$ for some positive constant c_1. Hence $u(t) \leq 0$ for t large enough, contradicting the fact that u is positive. If $u'(t) > 0$ for all t large, then

$$\int_{t'}^{t} \frac{u(t)}{t}dt \geq u(t')\int_{t'}^{t} \frac{u(s)}{s}ds \to \infty\,.$$

Thus u' has to be negative for some t large. **Q.E.D.**

The result in lemma 3.3 is almost sharp as we can get as close to $-n(n-1)/t^2$ as possible. For example, Let $R(g) = -n(n-1)$ let $f(t) = t\ln t$ for $t > 2$. We have

$$R = -\frac{1}{t^2}[n(n-1)\frac{(\ln t+1)^2}{(\ln t)^2} + \frac{2n}{\ln t} + \frac{n(n-1)}{(\ln t)^2}]\,. \tag{3.7}$$

We can show that it is not possible to conformally deform the metric

$$dt^2 + (t\ln t)^2 g$$

into a complete metric of nonnegative scalar curvature outside a compact set. Acturally we can show more. We first note that if

$$f(t)f''(t) \leq -c^2$$

for some positive constant c and for $t > t_o > 2$, then

$$f'(t) \leq f(t') - \int_{t'}^{t} \frac{c^2}{f(s)}ds\,,$$

where $t > t' > t_o$. As $f''(t) \leq 0$, we have $f(t) \leq Ct$ for all $t > t_o$, where C is a positive constant. Therefore

$$f'(t) \leq f(t') - (\frac{c}{C})^2 \int_{t'}^{t} \frac{1}{t}ds \to -\infty \quad \text{as} \quad t \to \infty\,.$$

Thus $f(t) \leq 0$ when t is large. As in theorem 1.12, we assume that the boundary of $\overline{M}$ is connected. And the case where $\partial\overline{M}$ has more than one connected component is similar.

Theorem 3.8. *For $n \geq 3$, let $\overline{M}$ be a compact $(n+1)$-manifold with boundary N. Suppose that N belongs to class (C) and g is a Riemannian metric on N with scalar curvature $R(g) \leq -\kappa^2$ for some positive constant κ. Let g' be a complete Riemannian metric on M with*

$$g'(t,x) = dt^2 + f(t)g(x) \quad \textit{on} \quad (2,\infty)\times N\,.$$

Suppose that $f(t)f''(t) \geq -c^2$ for all t large, where $c^2 = 2(\kappa^2-\delta)/(3n+1)$ and δ is constant such that $0 < \delta < \kappa^2$. Assume that either (i) $f(t) < Ct\ln t$ or (ii) $f(t) \geq Ct^\alpha$ with $\alpha > 1$, for t large, then there does not exist a complete Riemannian metric conformal to g' with nonnegative scalar curvature outside a compact set.

Proof. Let $g'' = u^{4/(n-1)}g'$ is a Riemannian metric conformal to g', where u is a positive smooth function. Assume that g'' is complete and the scalar curvature of g''

satisfies $R_{g''} \geq 0$ outside a compact set. The conformal scalar curvature equation [7] gives

$$\frac{\partial^2 u}{\partial t^2}+\frac{nf'(t)}{f(t)}\frac{\partial u}{\partial t}+\frac{1}{f^2(t)}\Delta_g u-\frac{1}{f^2(t)}\frac{n-1}{4n}\{R(g)-2nf(t)f''(t)-n(n-1)|f'(t)|^2\}u \leq 0\,.$$

for all $t \geq t'$. Fix a value $t \geq t'$ and integrate the above inequality with respect to the fixed Riemannian manifold (N, g), we have

$$\begin{aligned}
& \frac{\partial^2}{\partial^2 t}(\int_N u(x,t)dv_g) + \frac{nf'(t)}{f(t)}\frac{\partial}{\partial t}(\int_N u(x,t)dv_g) \\
+ & \frac{1}{f^2(t)}\frac{n-1}{4n}\{\kappa^2 + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\}(\int_N u(x,t)dv_g) \leq 0\,,
\end{aligned}$$

where we have used Green's theorem and the fact that $R_g \leq -\kappa^2$ for some positive constant κ. For $t \geq t'$, let

$$U(t) = \int_N u(x,t)dv_g\,.$$

Then we have

$$U'' + \frac{nf'(t)}{f(t)}U' + \frac{1}{f^2(t)}\frac{n-1}{4n}\{\kappa^2 + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\}U \leq 0\,. \tag{3.9}$$

For $t \geq t' > 0$, let $U(t) = f^\alpha(t)v(t)$, where α is a constant to be determined later. Then we have

$$\begin{aligned}
U'(t) &= \alpha f^{\alpha-1}(t)f'(t)v(t) + f^\alpha(t)v'(t)\,, \\
U''(t) &= \alpha(\alpha-1)f^{\alpha-2}(t)|f'(t)|^2 v(t) + \alpha f^{\alpha-1}(t)f''(t)v(t) + 2\alpha f^{\alpha-1}(t)f'(t)v'(t) + f^\alpha(t)v''(t)\,.
\end{aligned}$$

Substitute into (3.9) we have

$$\begin{aligned}
& f^{\alpha-2}(t)|f'(t)|^2 v(t)[\alpha(\alpha-1) + n\alpha + \frac{(n-1)^2}{4}] + f^\alpha(t)v''(t) \\
& +(n+2\alpha)f^{\alpha-1}(t)f'(t)v'(t) + f^{\alpha-2}(t)v(t)[\kappa^2 + (2n+\alpha)f(t)f''(t)] \leq 0
\end{aligned} \tag{3.10}$$

for all $t \geq t' > 0$. As

$$\alpha(\alpha-1) + n\alpha + \frac{(n-1)^2}{4} = [\alpha + \frac{n-1}{2}]^2 \geq 0$$

and $f(t)f''(t) \geq 2(-\kappa^2+\delta)/(3n+1)$, we have

$$f^\alpha(t)v''(t) + (n+2\alpha)f^{\alpha-1}(t)f'(t)v'(t) \leq -[\kappa^2 + (2n+\alpha)\frac{2(-\kappa^2+\delta)}{(3n+1)}]f^{\alpha-2}(t)v(t)\,.$$

Or

$$(f^{n+2\alpha}(t)v'(t))' \leq -[\kappa^2 + (2n+\alpha)\frac{2(-\kappa^2+\delta)}{(3n+1)}]f^{n+2\alpha-2}(t)v(t)\,. \tag{3.11}$$

Choose α such that $n + 2\alpha = 1$, that is, $\alpha = -(n-1)/2$, we have

$$(f(t)v'(t))' \le -\delta \frac{v(t)}{f(t)}\,.$$

Upon integration we have

$$f(t)v'(t) - f(t_o)v'(t_o) \le -\delta \int_{t_o}^{t} \frac{v(s)}{f(s)} ds\,, \tag{3.12}$$

where $t > t_o \ge t'$. If $v'(t) > 0$ for all $t \ge t'$, then we have

$$f(t)v'(t) \le f(t_o)v'(t_o) - \delta v(t_o) \int_{t_o}^{t} \frac{1}{f(s)} ds\,.$$

As $f(t) \le Ct \ln t$, for some positive constants C and for $t > t_o$, we have

$$\int_{t_o}^{t} \frac{1}{f(s)} ds \ge \frac{1}{C} \int_{t_o}^{t} \frac{1}{s \ln s} ds = \frac{1}{C}(\ln \ln t - \ln \ln t_o) \to \infty$$

as $t \to \infty$. That is, $v'(t) < 0$ when t is large. Thus we can find a t_o such that $v'(t_o) \le 0$. Hence

$$f(t)v'(t) \le -c'^2 \int_{t_o}^{t} \frac{v(s)}{s \ln s} ds\,,$$

that is $v'(t) \le 0$ for all $t \ge t_o$. Here c' is a positive constant. We have

$$f(t)v'(t) \le -c'^2 v(t) \int_{t_o}^{t} \frac{1}{s \ln s} ds\,. \tag{3.13}$$

For any positive constant $C > 0$, we can find $t'' > t_o$ such that for all $t \ge t''$, (3.13) gives

$$(\ln v(t))' \le -c'^2 \frac{\ln \ln t}{t \ln t}\,.$$

Therefore

$$\ln \frac{v(t)}{v(t_o)} \le -c''^2 (\ln \ln t)^2\,,$$

where c'' is a positive constant. Thus

$$v(t) \le \frac{C'}{(\ln t)^\beta}$$

for some positive constants C' and β and for $t > t_o$. Thus

$$U(t) \le \frac{C'}{f^{\frac{n-1}{2}}(t)(\ln t)^\beta} \le \frac{C''}{(t \ln t)^{\frac{n-1}{2}} (\ln t)^\beta}\,. \tag{3.14}$$

Since $n \ge 3$, we have $2/(n-1) \le 1$. By Hölder's inequality, we have

$$\int_N u^{\frac{2}{n-1}} dv_g \le [\mathrm{Vol}(N,g)]^{\frac{n-3}{n-1}} (\int_N u dv_g)^{\frac{2}{n-1}} \le C'' U^{\frac{2}{n-1}}(t) \le \frac{C'}{t (\ln t)^\gamma}\,,$$

where $\gamma > 1$ is a positive constant. Thus

$$\int_{t''}^{\infty} \int_N u^{\frac{2}{n-1}} dv_g dt < C' \int_{t''}^{\infty} \frac{dt}{t(\ln t)^{\gamma}} < \infty \,. \tag{3.15}$$

Hence we can find $x_o \in N$ such that

$$\int_{t''}^{\infty} u^{\frac{2}{n-1}}(x_o, t) dt < \infty \,.$$

Therefore the curve $\gamma(t) = (t, x_o)$ for $t \in (t'', \infty)$ has finite length in the Riemannian metric $g'' = u^{4/(n-1)} g'$, that is, the metric g'' is not complete. In case $f(t) > Ct^{\alpha}$ for $\alpha > 1$, then (3.12) gives

$$f(t) v'(t) \le C'$$

or

$$v'(t) \le \frac{C'}{t^{\alpha}}$$

Thus $v(t) \le C''$ for all $t > t_o$ and hence

$$U(t) \le \frac{C}{t^{\frac{(n-1)}{2}\alpha}} \,.$$

As $\alpha > 1$, we can conclude as above that the metric g'' is not complete. **Q.E.D.**

We note that functions of the type $f(t) = Ct^{\alpha}(\ln t)^{\beta}$ satisfy the conditon of theorem 3.8, where α, β, and $C > 0$ are constants, where if $\alpha = 1$, then $\beta \le 1$.

4. Scalar curvature of polar type Riemannian metrics

Given a compact n-manifold N and a constant $a > 0$, consider the following metric on $(a, \infty) \times N$:

$$\overline{g}(t, x) = dt^2 + f^2(t, x) g(x) \,, \tag{4.1}$$

where f is a positive smooth function on $(a, \infty) \times N$ and g is a fixed Riemannian metric on N. Let $R(f^2(t, \bullet) g)$ be the scalar curvature on N corresponding to the Riemannian metric $f^2(t, \bullet) g$, that is, the Riemannian metric conformal to g with conformal factor $f^2(t, \bullet)$, where t is treated as a constant. Let $\overline{R}$ be the scalar curvature of the Riemannian metric $\overline{g}$. We show that (c.f. [3])

$$\overline{R}(t, x) = R(f^2(t, \bullet) g)(t, x) - \frac{1}{f^2(t, x)} [2n f(t, x) \frac{\partial^2 f}{\partial t^2}(t, x) + n(n-1) |\frac{\partial f}{\partial t}(t, x)|^2] \,. \tag{4.2}$$

In particular, if f depends on t only, then

$$R(f^2(t, \bullet) g)(t, x) = \frac{1}{f^2(t)} R(g) \,,$$

which gives equation (1.2). In general, if $n \geq 3$ and we let $\mu(\bullet) = f^{\frac{n-2}{2}}(t,\bullet)$, then

$$R(f^2(t,\bullet)g) = c_n^{-1}\mu^{-\frac{n+2}{n-2}}[c_n R(g)\mu - \Delta_g \mu]\,, \tag{4.3}$$

where $c_n = (n-2)/[4(n-1)]$ [7].

Let $(x_1, x_3, ..., x_n)$ be a set of local coordinates for N and let $x_o = t$. Since $R_{0jk0} = R_{j00k}$, we have

$$\begin{aligned} \overline{R} &= \sum_{0\leq i,j,k,l\leq n} \overline{g}^{jl}\overline{g}^{ik}\overline{R}_{ijkl} \\ &= 2\sum_{1\leq j,k\leq n} \overline{g}^{jl}\overline{R}_{0j0l} + \sum_{1\leq i,j,k,l\leq n} \overline{g}^{jl}\overline{g}^{ik}\overline{R}_{ijkl}\,, \end{aligned} \tag{4.4}$$

where $\overline{R}_{ijkl}$ is the Riemannian curvature tensor for the metric $\overline{g}$ and is given by

$$\overline{R}_{ijkl} = \frac{1}{2}(\frac{\partial^2 \overline{g}_{jk}}{\partial x_i \partial x_l} + \frac{\partial^2 \overline{g}_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \overline{g}_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \overline{g}_{jl}}{\partial x_i \partial x_k}) + \sum_{0\leq a,b\leq n} \overline{g}_{ab}(\Gamma^b_{il}\Gamma^a_{jk} - \Gamma^b_{ik}\Gamma^a_{jl})\,.$$

Here

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{a=0}^{n} \overline{g}^{ia}(\frac{\partial \overline{g}_{aj}}{\partial x_k} + \frac{\partial \overline{g}_{ak}}{\partial x_j} - \frac{\partial \overline{g}_{ik}}{\partial x_a})\,, \quad 0 \leq i,j,k \leq n$$

are the Christoffel symbols for $\overline{g}$. Using

$$\begin{aligned} \Gamma^0_{jk} &= -f\frac{\partial f}{\partial t} g_{jk}\,, \quad 1 \leq i,j \leq n; \\ \Gamma^0_{0k} &= 0\,, \quad 0 \leq k \leq n; \\ \Gamma^i_{00} &= 0\,, \quad 0 \leq i \leq n; \\ \Gamma^i_{0k} &= \frac{1}{f}\frac{\partial f}{\partial t}\delta_{ik}\,, \quad 1 \leq i,k \leq n; \\ \Gamma^i_{jk} &= \frac{1}{2}\sum_{a=1}^{n} \overline{g}^{ia}(\frac{\partial \overline{g}_{aj}}{\partial x_k} + \frac{\partial \overline{g}_{ak}}{\partial x_j} - \frac{\partial \overline{g}_{ik}}{\partial x_a})\,, \quad 1 \leq i,j,k \leq n\,, \end{aligned} \tag{4.5}$$

we obtain $\overline{R}_{0000} = 0$ and for $1 \leq j,k \leq n$, we have

$$\sum_{1\leq j,k\leq n} \overline{g}^{jl}\overline{R}_{0j0l} = -\frac{n}{f}\frac{\partial^2 f}{\partial t^2}\,. \tag{4.6}$$

For $1 \leq i,j,k,l \leq n$, we have

$$\begin{aligned} \overline{R}_{ijkl} &= \frac{1}{2}(\frac{\partial^2 \overline{g}_{jk}}{\partial x_i \partial x_l} + \frac{\partial^2 \overline{g}_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \overline{g}_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \overline{g}_{jl}}{\partial x_i \partial x_k}) + \sum_{1\leq a,b\leq n} \overline{g}_{ab}(\Gamma^b_{il}\Gamma^a_{jk} - \Gamma^b_{ik}\Gamma^a_{jl}) \\ &\quad + \overline{g}_{00}(\Gamma^0_{il}\Gamma^0_{jk} - \Gamma^0_{ik}\Gamma^0_{jl}) \end{aligned}$$

The first two terms on the right hand side of the above equation has no derivatives with respect to t and by (4.5) it is equal to $R(f^2(t,\bullet)g)_{ijkl}$, the Riemannian curvature tensor for the Riemannian metric $f^2(t,\bullet)g$, for $1 \leq i,j,k,l$. And

$$(\Gamma^0_{il}\Gamma^0_{jk} - \Gamma^0_{ik}\Gamma^0_{jl}) = f^2|\frac{\partial f}{\partial t}|^2(g_{il}g_{jk} - g_{ik}g_{jl})\,.$$

Thus we have

$$
(4.7) \qquad \begin{aligned}
\sum_{1\le i,j,k,l\le n} \overline{g}^{jl}\overline{g}^{ik}\overline{R}_{ijkl} &= \sum_{1\le i,j,k,l\le n} \overline{g}^{jl}\overline{g}^{ik} R(f^2(t,\bullet)g)_{ijkl} \\
&\quad + \sum_{1\le i,j,k,l\le n} \overline{g}^{jl}\overline{g}^{ik} f^2 |\frac{\partial f}{\partial t}|^2 (g_{il}g_{jk} - g_{ik}g_{jl}) \\
&= R(f^2(t,\bullet)g) - \frac{n(n-1)}{f^2}|\frac{\partial f}{\partial t}|^2 .
\end{aligned}
$$

Substitute (4.6) and (4.7) into (4.4) we have obtained the desired formular. Using (4.4), the curve $\gamma(t) = (t, x)$ is a geodesic for $t > 2$, where $x \in N$. The metric $\overline{g}$ as defined in (4.1) is of polar type (c.f. [3]).

Theorem 4.8. *For $n \ge 3$, let g be a Riemannian metric on N with nonpositive total scalar curvature. Then the scalar curvature $\overline{R}$ of the Riemannian metric $\overline{g}$ in (4.1) cannot be uniformly positive for t large enough.*

Proof. Assume that $\overline{R} \ge \alpha^2$ for all $t > t'$ and $x \in N$, where α and $t' > a$ are positive constants. From (4.3) we have

$$
R(f^2(t,\bullet)g) = \frac{1}{f^2}[R(g) - c_n^{-1}\frac{\Delta_g \mu}{\mu}] .
$$

Substitute into equation (4.2) we have

$$
(4.9) \qquad \begin{aligned}
f^2\overline{R}(t,x) &= -c_n^{-1}\frac{\Delta_g\mu}{\mu} + R(g) - 2nf(t,x)\frac{\partial^2 f}{\partial t^2}(t,x) - n(n-1)|\frac{\partial f}{\partial t}(t,x)|^2 \\
&= -c_n^{-1}\frac{\Delta_g\mu}{\mu} - n\frac{\partial^2(f^2)}{\partial t^2} - n(n-3)|\frac{\partial f}{\partial t}|^2 ,
\end{aligned}
$$

where we have used the formular

$$
-2nf\frac{\partial^2 f}{\partial t^2} = -n\frac{\partial^2(f^2)}{\partial t^2} + 2n|\frac{\partial f}{\partial t}|^2 .
$$

Fix a $t > t'$ and integrate both sides of (4.9) with respect to (N, g), we have

$$
(4.10) \qquad \int_N f^2\overline{R}dv_g = -c_n^{-1}\int_N \frac{|\nabla\mu|^2}{\mu^2}dv_g + \int_N R(g)dv_g - n\int_N \frac{\partial^2(f^2)}{\partial t^2}dv_g - n(n-3)\int_N |\frac{\partial f}{\partial t}|^2 dv_g ,
$$

where we have used Green's identity. Using the fact that $n \ge 3$, $\overline{R} \ge \alpha^2$ for $t > t'$ and

$$
\int_N R(g)dv_g \le 0 ,
$$

we have

$$
a^2 F \le -n\frac{d^2 F}{dt^2} ,
$$

where

$$F(t) = \int_N f^2 dv_g \,.$$

Or

$$F''(t) \le -b^2 F(t) \quad \text{for} \quad t > t' \,, \tag{4.11}$$

where $b = \alpha/\sqrt{n}$ is a positive constant. If there is $t_o > t'$ such that $F'(t_o) < 0$, then integrating both sides of (4.11) gives

$$F'(t) \le F'(t_o) - b^2 \int_{t_o}^{t} F(s) ds \quad \text{for} \quad t > t_o \,. \tag{4.12}$$

Thus $F(t) \le 0$ for some t large enough, contradicting that f is a positive function. Therefore $F'(t) \ge 0$ for all $t > t'$. Then (4.11) gives

$$F'(t) \le F'(t_o) - b^2 F(t_o) \int_{t_o}^{t} ds \to -\infty \quad \text{as} \quad t \to \infty \,,$$

which contradicts $F'(t) \ge 0$ for all $t > t'$. **Q.E.D.**

Conbining the proof of lemma 1.8 and the above theorem 4.8, it can be shown that the scalar curvature $\overline{R}$ of $\overline{g}$ cannot decay to zero too slowly. More precisely, $\overline{R}$ cannot be bigger than or equal to

$$\frac{cn}{4t^2}$$

for all $t > t' > a$ and $x \in N$, where $c > 1$ and t' are constants. For if

$$\overline{R} \ge \frac{cn}{4t^2} \,,$$

then (4.10) gives

$$t^2 F'' + \frac{c}{4} F \le 0$$

for all $t > t'$. The proof of lemma 1.8 shows that $F(t) = 0$ for some t large. On the other hand it is known that $F(t) > 0$ for all $t > a$.

Theorem 4.13. *For $n \ge 3$, let g be a Riemannian metric on N with negative total scalar curvature. Then the scalar curvature $\overline{R}$ of the Riemannian metric $\overline{g}$ in (4.1) cannot be bigger than $-c/t^2$ for some constant $c < 2n$ and for all $t > t' > 2$ and $x \in N$. In particular, $\overline{R}$ cannot be nonnegative for all t large.*

Proof. Assume that

$$\int_N R(g) dv_g \le -b^2$$

and

$$\overline{R}(t, x) \ge -\frac{c}{t^2} \quad \text{for} \quad t > t' \quad \text{and} \quad x \in N \,,$$

where $c < 2n$ and b is a positive constant. Then (4.10) gives

$$-\frac{c}{t^2}F(t) \le -b^2 - nF''(t) \quad \text{for } \ t > t' \,,$$

or

$$F''(t) \le -\frac{b^2}{n} + \frac{c'}{t^2}F(t) \quad \text{for } \ t > t' \,, \tag{4.14}$$

where $c' = c/n < 2$ is a constant. If $c' \le 0$, then (4.14) shows that $F(t) \le 0$ for t large enough. So we may assume that $c' > 0$. From (4.14) we have

$$\frac{F''(t)}{F(t)} \le \frac{c'}{t^2} \quad \text{for } \ t > t' \,.$$

It follows from the proof of lemma 3.3 that

$$F(t) \le Ct^{\epsilon} \quad \text{for } \ t > t_o > t' \,, \tag{4.15}$$

where C is a positive constant and $\epsilon > 1$ is a positive constant such that $\epsilon(\epsilon - 1) = c'$. In particular, $\epsilon < 2$. Then (4.14) and (4.15) imply that $F''(t) \le -b^2/(2n)$ for all t large enough. Thus $F(t) \le 0$ when t is large enough. **Q.E.D.**

The Laplacian for the metric $\overline{g}(t,x) = dt^2 + f^2(t,x)g(x)$ is given by

$$\Delta_{\overline{g}} u = \frac{\partial^2 u}{\partial t^2} + \frac{n}{f}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t} + \frac{n-2}{f^3} < \nabla_g f \,, \nabla_g u >_g + \frac{1}{f^2}\Delta_g u \tag{4.16}$$

for $u \in C^\infty((a\,,\infty) \times N)$. Suppose that $u(t,x)$ is a positive smooth function. The scalar curvature R_c of the Riemannian metric

$$u^{\frac{4}{n-1}}(t,x)\overline{g} = u^{\frac{4}{n-1}}(t,x)[dt^2 + f^2(t,x)g(x)]$$

is given by the following:

$$\Delta_{\overline{g}}\, u - c_{n+1}R_{\overline{g}}u + c_{n+1}R_c u^{\frac{n+3}{n-1}} = 0 \,, \tag{4.17}$$

where $c_{n+1} = (n-1)/(4n)$.

Theorem 4.18. *For $n \ge 3$, let $\overline{M}$ be a compact $(n+1)$-manifold with boundary N. Suppose that N is connected and g is a Riemannian metric on N. Suppose that f is a positive smooth function on $(2,\infty) \times N$ such that*

$$|\frac{\partial f}{\partial t}| \le C_1\frac{f}{t} \quad \text{and} \quad |\frac{\partial^2 f}{\partial t^2}| \le C_2\frac{f}{t^2} \,, \tag{4.19}$$

for all $t > t' > 0$ and $x \in N$. Let $\overline{g}$ be a complete Riemannian metric on M such that $\overline{g}(t,x) = dt^2 + f^2(t,x)g(x)$ on $(2,\infty) \times N$. Assume that $\overline{g}$ has scalar curvature

$R_{\overline{g}} \le -b^2$ for all $t > t'$ and $x \in N$, where $t' > 2$, C_1, C_2 and b are positive constants. Then for any positive smooth function u on $(a, \infty) \times N$ with

$$(4.20) \qquad |\frac{\partial u}{\partial t}| \le Cu \quad \text{for all} \quad t > t_1 \quad \text{and} \quad x \in N\,,$$

where $t_1 > 2$ and C are positive constant, if $u^{\frac{4}{n-1}}\,\overline{g}$ is a complete Riemannian metric , then the scalar curvature R_c of the conformal metric $u^{\frac{4}{n-1}}\,\overline{g}$ cannot not be nonnegative for all t large .

Proof. Assume that the scalar curvature of the Riemannian metric $u^{\frac{4}{n-1}}\,\overline{g}$ is nonnegative for all $t > t_2$ and $x \in N$, where $t_2 > 2$ is a constant. If $t_o = \max\{t'\,, t_1\,, t_2\,,\}$, then for $t > t_o$, the scalar curvature equation (4.17) togather with (4.16) and (4.17) give

$$(4.21) \qquad \frac{\partial^2 u}{\partial t^2} + \frac{n}{f}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t} + \frac{n-2}{f^3} < \bigtriangledown_g f\,, \bigtriangledown_g u >_g + \frac{1}{f^2}\Delta_g u + c^2 u \le 0\,,$$

where $c^2 = c_{n+1} b^2$ is a positive constant. Multiple (4.21) by f^n we obtain

$$(4.22) \qquad f^n \frac{\partial^2 u}{\partial t^2} + n f^{n-1}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t} + < \bigtriangledown_g f^{n-2}\,, \bigtriangledown_g u >_g + f^{n-2}\Delta_g u + c^2 f^n u \le 0\,.$$

Fix a $t > t_o$ and integrate (4.22) with respect to the Riemannian metric g and apply Green's identity, we have

$$\int_N f^n \frac{\partial^2 u}{\partial t^2} dv_g + \int_N n f^{n-1}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t} dv_g + c^2 \int_N f^n u dv_g \le 0$$

for all $t > t_o$, or

$$(4.23) \qquad \frac{d}{dt}\Big(\int_N f^n \frac{\partial u}{\partial t} dv_g\Big) \le -c^2 \mathcal{F}(t)\,,$$

where

$$\mathcal{F}(t) = \int_N f^n u dv_g > 0 \quad \text{for} \quad t > t_o\,.$$

We have

$$\begin{aligned}
\frac{d\mathcal{F}}{dt} &= \int_N n f^{n-1}\frac{\partial f}{\partial t} u dv_g + \int_N f^n \frac{\partial u}{\partial t} dv_g\,, \\
\frac{d^2\mathcal{F}}{dt^2} &= \int_N n(n-1) f^{n-2} |\frac{\partial f}{\partial t}|^2 u dv_g + \int_N n f^{n-1}\frac{\partial^2 f}{\partial t^2} u dv_g + \int_N n f^{n-1}\frac{\partial f}{\partial t}\frac{\partial u}{\partial t} dv_g \\
&\quad + \frac{d}{dt}\Big(\int_N f^n \frac{\partial u}{\partial t} dv_g\Big)\,.
\end{aligned}$$

Using (4.19), (4.20) and (4.23) we have

$$(4.24) \qquad \frac{d^2\mathcal{F}}{dt^2}(t) \le \frac{n(n-1)C_1^2}{t^2}\mathcal{F}(t) + \frac{nC_2}{t^2}\mathcal{F}(t) + \frac{nCC_1}{t}\mathcal{F}(t) - c^2\mathcal{F}(t)$$

for $t > t_o$. Thus we can find positive constants $\overline{t} > t_o$ and c' such that

$$\mathcal{F}''(t) \leq -c'^2 \mathcal{F}(t) \quad \text{for all} \quad t > \overline{t}\,. \tag{4.25}$$

As in the proof of theorem 4.8, (4.25) implies that $\mathcal{F}(t) \leq 0$ when t is large enough. **Q.E.D.**

It follows as in theorem 4.8 that we can relax the condition on the scalar curvature of $\overline{g}$ to

$$R_{\overline{g}} \leq -\frac{b^2}{t^\alpha}$$

for all $t > t'$ and $x \in N$, where $t' > a$, b and $\alpha < 1$ are positive constants. Then (4.24) gives

$$\frac{d^2\mathcal{F}}{dt^2}(t) \leq \frac{n(n-1)C_1^2}{t^2}\mathcal{F}(t) + \frac{nC_2}{t^2}\mathcal{F}(t) + \frac{nCC_1}{t}\mathcal{F}(t) - \frac{c^2}{t^\alpha}\mathcal{F}(t)\,.$$

For t large enough, since $\alpha < 1$, we have

$$\mathcal{F}''(t) \leq -\frac{C'}{t^\alpha}\mathcal{F}(t) \leq -\frac{c'}{4}\frac{1}{t^2}\mathcal{F}(t)\,,$$

where C' and $c' > 1$ are positive constants. The proof of lemma 1.8 implies that $\mathcal{F}(t) = 0$ for some large t.

Appendix

Proposition A.1. *For $n \geq 2$, let $\overline{M}$ be a compact $(n+1)$-manifold with boundary ∂M and interior M. Given any smooth function $R \in C^\infty(\overline{M})$, there is a Riemannian metric g (non-complete if $\partial M \neq \emptyset$) defined on $\overline{M}$ such that R is the scalar curvature of g in M.*

Proof. If $\overline{M}$ is a compact manifold with boundary, then the double of $\overline{M}$, defined by

$$2\overline{M} = (\overline{M} \times \{1\} \cup \overline{M} \times \{2\}\,)/\partial M\,,$$

can be given a C^∞ structure as a compact manifold without boundary, such that the inclusions

$$\begin{aligned} i_k : \overline{M} \quad &\hookrightarrow \quad 2\overline{M} \\ x \quad &\rightarrow \quad (x,k)/\sim\,, \qquad k = 1,2\,, \end{aligned}$$

are diffeomorphisms onto their range [9]. By Seeley's extension theorem [9], any smooth function R defined on $\overline{M} \cong i_1(\overline{M})$ can be extended to a smooth function R'

on $2\overline{M}$. We can modify the function R' on $i_2(M)$ so that it is negative somewhere there. Then the classification theorem of Kazdan and Warner implies that there is a Riemannian metric g' on $2\overline{M}$ such that the scalar curvature of g' is the extended function R'. We can take g to be the restriction of g' on $i_1(M)$. The Riemannian metric g is not complete if ∂M is nonempty. **Q.E.D.**

References

[1] P. Aviles & R. McOwen, *Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds,* J. Diff. Geom. **27** (1988), 225-239.

[2] A. Besse, *Einstein Manifolds,* Springer-Verlag, Berlin-New York, 1987.

[3] J. Bland & M. Kalka, *Negative scalar curvature metrics on non-compact manifolds,* Trans. A.M.S. **316** (1989), 433-446.

[4] F. Dobarro & E. Lami Dozo, *Scalar curvature and warped products of Riemannian manifolds,* Trans. A.M.S. **303** (1987), 161-168.

[5] M. Gromov & H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds,* Publ. Math. I.H.E.S. **58** (1983), 295-408.

[6] H.B. Lawson & M. Michelsohn, *Spin Geometry,* Princeton University Press, Princeton, 1989.

[7] M.C. Leung, *Conformal scalar curvature equations on complete manifolds,* Comm. in P.D.E. **20** (1995), 367-417.

[8] Y. Li & W. Ni, *On conformal scalar curvature equations in* $\mathbf{R}^n$*,* Duke Math. J. **57** (1988), 895-924.

[9] R.B. Melrose, *Differential Analysis on Manifolds with Corners,* M.I.T. lecture notes, 1988.

[10] A. Ratto, M. Rigoli & L. Véron, *Scalar curvature and conformal deformation of hyperbolic space,* J. Functional Analysis **121** (1994), 15-77.